HALF EIGENVALUES AND THE FUČÍK SPECTRUM OF MULTI-POINT, BOUNDARY VALUE PROBLEMS

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ABSTRACT. We consider the nonlinear boundary value problem consisting of the equation

$$-u'' = f(u) + h, \quad \text{a.e. on } (-1,1), \tag{1}$$

where $h \in L^1(-1,1)$, together with the multi-point, Dirichlet-type boundary conditions

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{2}$$

where $m^\pm \geqslant 1$ are integers, $\alpha^\pm = (\alpha_1^\pm, \ldots, \alpha_m^\pm) \in [0,1)^{m^\pm}$, $\eta^\pm \in (-1,1)^{m^\pm}$, and we suppose that

$$\sum_{i=1}^{m^\pm} \alpha_i^\pm < 1.$$

We also suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and

$$0 < f_{\pm\infty} := \lim_{s \to \pm\infty} \frac{f(s)}{s} < \infty$$

(we assume that these limits exist). We allow $f_\infty \neq f_{-\infty}$ — such a nonlinearity f is said to be *jumping*.

Related to (1) is the equation

$$-u'' = \lambda(au^+ - bu^-), \quad \text{on } (-1,1), \tag{3}$$

where λ, a, $b > 0$, and $u^\pm(x) = \max\{\pm u(x), 0\}$ for $x \in [-1,1]$. The problem (2)-(3) is 'positively-homogeneous' and jumping. Regarding a, b as fixed, values of $\lambda = \lambda(a,b)$ for which (2)-(3) has a non-trivial solution u will be called *half-eigenvalues*, while the corresponding solutions u will be called *half-eigenfunctions*.

We show that a sequence of half-eigenvalues exists, the corresponding half-eigenfunctions having specified nodal properties, and we obtain certain spectral and degree theoretic properties of the set of half-eigenvalues. These properties lead to solvability and non-solvability results for the problem (1)-(2). The set of half-eigenvalues is closely related to the 'Fučík spectrum' of the problem, which we briefly describe. Equivalent solvability and non-solvability results for (1)-(2) are obtained from either the half-eigenvalue or the Fučík spectrum approach.

1. INTRODUCTION

We consider the nonlinear boundary value problem consisting of the equation

$$-u'' = f(u) + h, \quad \text{a.e. on } (-1,1), \tag{1.1}$$

where $h \in L^1(-1,1)$, together with the multi-point, Dirichlet-type boundary conditions

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{1.2}$$

where $m^\pm \geqslant 1$ are integers, $\alpha^\pm = (\alpha_1^\pm, \dots, \alpha_m^\pm) \in \mathbb{R}^{m^\pm}$, $\eta^\pm \in (-1,1)^{m^\pm}$. We suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and

$$0 < f_{\pm\infty} := \lim_{s \to \pm\infty} \frac{f(s)}{s} < \infty \tag{1.3}$$

(we assume that these limits exist). We allow $f_\infty \neq f_{-\infty}$ in (1.3) — such a nonlinearity f is said to be *jumping*.

We will require some further notation, and conditions, for the coefficients $\alpha^\pm$, which we now describe. For any integer $m \geqslant 1$ and any point $\alpha \in \mathbb{R}^m$, the notation $\alpha = 0$, $\alpha \geqslant 0$, $\alpha > 0$ will mean $\alpha_i = 0$, $\alpha_i \geqslant 0$, $\alpha_i > 0$, for $i = 1, \dots, m$, respectively; $\mathcal{A}^m$ will denote the set of $\alpha \in \mathbb{R}^m$ satisfying

$$\sum_{i=1}^m |\alpha_i| < 1, \tag{1.4}$$

while $\mathcal{A}_+^m$ will denote the set of $\alpha \in \mathcal{A}^m$ satisfying $\alpha \geqslant 0$. Throughout the paper we will suppose that $\alpha^\pm \in \mathcal{A}_+^{m^\pm}$.

Related to (1.1) is the equation

$$-u'' = au^+ - bu^-, \quad \text{on } (-1,1), \tag{1.5}$$

where a, $b > 0$, and $u^\pm(x) = \max\{\pm u(x), 0\}$ for $x \in [-1,1]$. Putting $a = f_\infty$, $b = f_{-\infty}$, and $h = 0$, we can regard (1.5) as a limiting form of (1.1) as $|u| \to \infty$ (this will be made more precise below). However, we will consider (1.5), and variants of this equation, in its own right with general values of a and b (see Remark 3.3 (ii) below for why we assume that a, $b > 0$).

The boundary value problem (1.2), (1.5) has a positively-homogeneous jumping nonlinearity, in the sense that if u is a solution then tu is also a solution for all $t \geqslant 0$. We can now define the *Fučík spectrum* of (1.2), (1.5), to be the set

$$\Sigma_F := \{(a,b) \in \mathbb{R}^2 : \text{ (1.2), (1.5) has a non-trivial solution } u\}.$$

The Fučík spectrum of problems with separated boundary conditions has been used extensively to derive criteria for the solvability, or non-solvability, of the general nonlinear problem (1.1), (1.2). Such criteria have usually been described in terms of the location of the point $(f_\infty, f_{-\infty})$ in $\mathbb{R}^2$ relative to the spectrum Σ_F. See, for example, [4, 7, 11] and the references therein (there are over 80 papers on Mathscinet with 'Fučík spectrum' in the title).

An alternative, real-valued, spectrum, which also yields solvability criteria for (1.1), (1.2), can be defined by considering the equation

$$-u'' = \lambda(au^+ - bu^-), \quad \text{on } (-1,1), \tag{1.6}$$

with a spectral parameter $\lambda > 0$. The problem (1.2), (1.6) again has a positively-homogeneous jumping nonlinearity. Regarding a, b as fixed, we will say that any λ for which (1.2), (1.6) has a non-trivial solution u is a *half-eigenvalue*, and u is a *half-eigenfunction*, and we define a corresponding spectrum to be the set

$$\Sigma_H = \Sigma_H(a,b) := \{\lambda \in \mathbb{R} : \text{ (1.2), (1.6) has a non-trivial solution } u\}.$$

For problems with separated boundary conditions the half-eigenvalue spectrum has also been extensively investigated, see for example [11, 12], and the references therein, and a detailed comparison of the Fučík and half-eigenvalues approaches to the solvability of (1.1), (1.2) is given in [12].

Of course, in the current setting (with a, b constant) these spectra are closely related, since a point $(a, b) \in \Sigma_F$ if and only if $1 \in \Sigma_H(a, b)$. However, when the coefficients a, b are variable (which one would naturally consider when f, and the limits $f_{\pm\infty}$, depend on x) it is not so clear how to even define the Fučík spectrum, and it is more difficult to obtain solvability criteria for the general problem (1.1), (1.2). In this case such criteria are usually given in terms of the location of the set of points $\{(f_\infty(x), f_{-\infty}(x)) : x \in (-1, 1)\} \subset \mathbb{R}^2$, relative to the constant coefficient Fučík spectrum Σ_F, see [12] for more details. On the other hand, when a, b are variable the spectrum $\Sigma_H(a, b)$ can be defined just as above, and it is shown in [12] that in this case the half-eigenvalue approach can yield stronger solvability results than the Fučík spectrum approach.

In our discussion of the multi-point problem here we will only consider the constant coefficient problem, so the two approaches are, in principle, equivalent. Despite this, we will use the half-eigenvalue approach since our method of obtaining the spectrum, and its properties, will rely heavily on having a single parameter $\lambda \in \mathbb{R}$, rather than a point $(a, b) \in \mathbb{R}^2$.

Remark 1.1. The papers [11, 12] actually use half-eigenvalues defined by equations of the form

$$-u'' = au^+ - bu^- + \lambda u, \quad \text{on } (-1, 1), \tag{1.7}$$

rather than (1.6). However, for the separated problem there is little difficulty in converting the results of [11, 12] to the setting of (1.6). Somewhat surprisingly, we find that this is not true in the multi-point setting. In fact, we find that it is considerably easier to deal with the half-eigenvalue formulation (1.6) rather than (1.7), which is why we use (1.6) here. Indeed, the paper [14] considered (1.7) together with the 3-point boundary condition

$$u(-1) = 0, \quad u(1) = \alpha u(\eta)$$

(in the present notation), and encountered considerable technical difficulties. The results we obtain here will generalise the results of [14] to the much more general multi-point conditions (1.2), while avoiding many of the difficulties encountered in [14].

Remark 1.2. When $\alpha^\pm = 0$ the boundary conditions (1.2) reduce to the standard, separated Dirichlet conditions at $x = \pm 1$, so we have termed the conditions (1.2) 'Dirichlet-type'. Similar results to those below can also be obtained for multi-point 'Neumann-type' conditions, or a mixture of these, see [9] or [16] (the Neumann-type problem is slightly more complicated because a 'multi-point operator' Δ, which we introduce in Section 2.1 below, is non-singular since constant functions lie in its kernel — however, this can be dealt with in the manner described in [16]). More general nonlocal boundary conditions can also be considered, as described in Section 1.1 of [9]. For brevity we will not consider any of these cases further here — given the methods here, the extensions to these other cases is relatively straightforward.

We conclude this section with a brief description of the contents of the paper. Some preliminary results are described in Section 2. In Section 3 we state (without proofs) our main results on the spectral theoretic properties of the half-eigenvalue problem, and on solvability and non-solvability conditions for a corresponding inhomogeneous form of (1.2), (1.6) (the solvability conditions are expressed in terms

of the half-eigenvalues). These results are similar to those obtained in [11] for separated boundary conditions. We also give some examples which show that our assumption that the coefficients $\alpha^{\pm} \in \mathcal{A}_+^{m^{\pm}}$ is optimal, and cannot be relaxed to the condition $\alpha^{\pm} \in \mathcal{A}^{m^{\pm}}$, which is known to suffice for the linear, multi-point eigenvalue problem (see [15]). These results also enable us to easily construct the Fučík spectrum Σ_F, and we give a brief description of this in Section 3.3. We then begin the proofs. In Section 4 we prove an existence and uniqueness result for a problem consisting of equation (1.2) together with a single, multi-point, boundary condition (this problem could be regarded as a multi-point analogue of the usual initial value problem for equation (1.2)). As usual, the uniqueness result for this multi-point, 'initial value problem' then implies the simplicity of the half-eigenvalues (in a suitable sense). The results stated in Section 3 are then proved in Sections 5 and 6. In Section 7 we extend the solvability and non-solvability results of Section 3 to the more general problem (1.1), (1.2), and in Section 8 we obtain a global bifurcation theorem, and nodal solutions of this problem.

2. Preliminary definitions and results

2.1. **Function spaces and notation.** For any integer $n \geqslant 0$, $C^n[-1,1]$ will denote the usual Banach space of n-times continuously differentiable functions on $[-1,1]$, with the usual sup-type norm, denoted by $|\cdot|_n$ (all function spaces are taken to be real). Let

$$X := \{u \in C^2[-1,1] : u \text{ satisfies } (1.2)\}, \quad Y := C^0[-1,1],$$

with the norms $|\cdot|_2$ and $|\cdot|_0$ respectively. We define a bounded linear operator $\Delta : X \to Y$ by

$$\Delta u := -u'', \quad u \in X.$$

This operator has a bounded inverse $\Delta^{-1} : Y \to X$, see [15, Theorem 3.1]. The following spaces will also be useful

$$\widetilde{X} := \{u \in W^{2,1}(-1,1) : u \text{ satisfies } (1.2)\}, \quad \widetilde{Y} := L^1(-1,1),$$

where $W^{2,1}(-1,1)$ denotes the usual Sobolev space of functions with second order derivatives in $L^1(-1,1)$ ($L^1(-1,1)$ and $W^{2,1}(-1,1)$ will be endowed with their usual norms). It can readily be checked that Δ extends to a bounded, invertible operator $\widetilde{\Delta} : \widetilde{X} \to \widetilde{Y}$, see [15, Remark 3.4].

Up to now we have regarded $\alpha^{\pm}$, $\eta^{\pm}$, as constant, and omitted them from the notation, for example, for the operator Δ. However, in some of the discussion below it will be convenient to regard some, or all, of these as variable, and we will then indicate the dependence on these variables in the obvious manner. In particular, we let $\boldsymbol{\alpha} := (\alpha^-, \alpha^+)$ and $\mathcal{A}_+ := \mathcal{A}_+^{m^-} \times \mathcal{A}_+^{m^+}$, $\boldsymbol{\eta} := (\eta^-, \eta^+)$, and we will write, for example, $\Delta^{-1}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \mathcal{A}_+$ (note that X also depends on $\boldsymbol{\alpha}$, but Y does not).

2.2. **Nodal properties.** We now introduce some notation to describe the nodal properties of solutions of (1.1) or (1.5). Firstly, for any C^1 function u, if $u(x_0) = 0$ then x_0 is a *simple* zero of u if $u'(x_0) \neq 0$. Now, for any integer $k \geqslant 1$ and any $\nu \in \{\pm\}$, we define $T_{k,\nu} \subset X$ to be the set of functions $u \in X$ satisfying the following conditions:

T-(a) $u'(\pm 1) \neq 0$ and $\nu u'(-1) > 0$;
T-(b) u' has only simple zeros in $(-1,1)$, and has exactly k such zeros;

T-(c) u has a zero strictly between each consecutive zero of u'.

We also define $T_k := T_k^+ \cup T_k^-$.

Remarks 2.1. (i) If $u \in T_{k,\nu}$ then u has exactly one zero between each consecutive zero of u', and all zeros of u are simple. Thus, u has at least $k-1$ zeros in $(-1,1)$, and at most k zeros in $(-1,1]$.
(ii) The sets $T_{k,\nu}$ are open in X and disjoint.
(iii) The sets $T_{k,\nu}$ were introduced in [13]. Similar, but slightly different, sets have been used to describe the nodal properties of solutions in the separated boundary condition case, but it is shown in [13] that the sets $T_{k,\nu}$ are more suitable for the multi-point problem.
(iv) The symbols $\pm$ are used in two different contexts in this paper: (a) they refer to the end points ±1 at which the boundary conditions hold (cf. (1.2)); (b) they refer to the sign properties of the nodal solutions u, in particular, through part T-(a) of the above definition of the sets $T_{k,\nu}$. To attempt to keep the distinction between these usages clearer, we use superscript $\pm$ to denote the boundary condition context, and subscript $\pm$ to denote the sign context (except in the usage $u^\pm$ for the positive and negative parts of u, as in equation (1.6) — this usage seems too well-established in the literature to change it here).

3. Half-eigenvalues and solvability properties

3.1. **Half-eigenvalues.** In this section we consider the half-eigenvalue problem (1.2), (1.6), which we rewrite in the form

$$-\Delta u = \lambda(au^+ - bu^-), \quad u \in X. \tag{3.1}$$

General results on half-eigenvalues, and associated solvability and degree theoretic results, are described in [1] and [11] for Sturm-Liouville problems with separated boundary conditions. In this section we describe similar results for the multi-point problem.

Theorem 3.1. *Suppose that $a, b > 0$ and $\boldsymbol{\alpha} \in \mathcal{A}_+$. For each $k \geqslant 1$ and $\nu \in \{\pm\}$ there is exactly one half-eigenvalue $\lambda_{k,\nu} = \lambda_{k,\nu}(a,b) > 0$ of (3.1) with a half-eigenfunction $u_{k,\nu} = u_{k,\nu}(a,b) \in T_{k,\nu}$, and there are no other half-eigenvalues. The sequence of half-eigenvalues is strictly increasing, in the sense that*

$$k' > k \implies \lambda_{k',\nu'} > \lambda_{k,\nu}, \quad \textit{for each } \nu',\, \nu \in \{\pm\}. \tag{3.2}$$

In addition, $\pm u_{1,\pm}$ is strictly positive on $(-1,1)$, and $\lim_{k\to\infty} \lambda_{k,\pm} = \infty$.

Naturally, the half-eigenvalues in Theorem 3.1 depend on all the parameters in the problem (3.1). The following result shows that this dependence is continuous.

Corollary 3.2. *The half-eigenvalues $\lambda_{k,\pm}$, $k \geqslant 1$, are C^1 functions of the variables $(a, b, \boldsymbol{\alpha}, \boldsymbol{\eta})$ in $\mathbb{R}^2 \times \mathcal{A}_+ \times (-1,1)^{m^-+m^+}$.*

Remarks 3.3. It can easily be seen, from the form of the differential equation (1.6) when u is positive or negative, that if a (non-trivial) solution u changes sign at least 3 times then a, b, λ must all be non-zero and have the same sign, so Theorem 3.1 cannot hold unless this is true. Thus, without loss of generality, we assume here that a, b, λ are all positive.

Remarks 3.4. For the separated half-eigenvalue problem ($a \neq b$, $\boldsymbol{\alpha} = 0$), similar results to those of Theorem 3.1 are proved in [1, Theorem 2], [2, Theorem 4] and [11, Theorem 5.1], although these papers mainly consider half-eigenvalue problems of the form

$$-\Delta u = au^+ - bu^- + \lambda u \tag{3.3}$$

with variable coefficients a, b (recall Remark 1.1).

Remark 3.5 (Eigenvalues). When $a = b$ (with $\boldsymbol{\alpha} \neq \mathbf{0}$) the problem (3.1) reduces to the linear multi-point eigenvalue problem

$$-\Delta u = \lambda a u, \quad u \in X. \tag{3.4}$$

The spectral properties of (3.4), with $a = 1$, are obtained in [15, Theorem 3.1], and it is clear that if we denote the eigenvalues and eigenfunctions of this problem by λ_k and u_k, $k \geqslant 1$, then $\lambda_{k,\pm}(a, a) = \lambda_k/a$ and we may suppose that $u_{k,\pm}(a, a) = \pm u_k$. Thus Theorem 3.1 above extends the linear eigenvalue results in [15, Theorem 3.1] to the half-eigenvalue problem (however, [15] also deals with the p-Laplacian eigenvalue problem, and the results in [15] are valid for all $\alpha^\pm \in \mathcal{A}^{m^\pm}$).

It is shown in [15] that for the eigenvalue problem (with $a = b = 1$) Theorem 3.1 holds for all $\boldsymbol{\alpha} \in \mathcal{A}$, but need not be true if $\boldsymbol{\alpha} \notin \mathcal{A}$. The following examples show that for the half-eigenvalue problem Theorem 3.1 need not be true if $\boldsymbol{\alpha} \in \mathcal{A} \setminus \mathcal{A}_+$, that is, if $\alpha^\pm$ satisfy (1.4) but some of the coefficients $\alpha_i^\pm$ are negative. Each of these examples have a Dirichlet condition at -1, so the corresponding solutions can easily be sketched, and the truth of the assertions should then be clear.

Example 3.6. The problem

$$\begin{gathered} -u'' = \left(\tfrac{2\pi}{3}\right)^2 u^+ - (\pi)^2 u^- \\ u(-1) = 0, \quad u(1) = -\tfrac{2}{3}u(-\tfrac{1}{4}). \end{gathered}$$

has a half-eigenfunction $u \in \partial T_{1,+}$ given by

$$u(x) := \begin{cases} \frac{3}{2\pi} \sin \frac{2\pi}{3}(x+1), & x \in (-1, \frac{1}{2}), \\ -\frac{1}{\pi} \cos \pi(x-1), & x \in (\frac{1}{2}, 1). \end{cases}$$

Example 3.7. If $\delta > 0$ is sufficiently small then the problem

$$\begin{gathered} -u'' = \left(\tfrac{\pi}{2} + \delta\right)^2 u^+ - \left(\tfrac{1}{\delta}\right)^2 u^- \\ u(-1) = 0, \quad u(1) = -\tfrac{1}{2}u(0). \end{gathered}$$

has no half-eigenfunction $u \in T_{2,+}$.

Remark 3.8. It will be seen that the proofs of Theorems 3.1 and 8.1 below rely on the fact that eigenfunctions cannot lie in the boundaries of the sets $T_{k,\nu}$ (so that the nodal properties of the eigenfunctions are preserved as parameters are varied). It will be shown that this is true when $\boldsymbol{\alpha} \in \mathcal{A}_+$, but Example 3.6 shows that it need not be true when $\boldsymbol{\alpha} \in \mathcal{A} \setminus \mathcal{A}_+$.

3.2. **Solvability properties.** In addition to eigenvalues, linear spectral theory is also concerned with the solvability of inhomogeneous problems. Accordingly, we will consider the solvability of the inhomogeneous equation

$$-\widetilde{\Delta}u = \lambda(au^{+} - bu^{-}) + h, \quad u \in \widetilde{X}, \tag{3.5}$$

for general functions $h \in \widetilde{Y}$, when λ is not a half-eigenvalue (if $h \in Y$, then we would use the operator Δ and search for solutions $u \in X$, in the same manner).

Clearly, (3.5) is equivalent to the equation

$$R_\lambda(u) := u + \lambda\widetilde{\Delta}^{-1}(au^{+} - bu^{-}) = -\widetilde{\Delta}^{-1}h, \quad u \in \widetilde{X}, \tag{3.6}$$

and the operator $R_\lambda : \widetilde{X} \to \widetilde{X}$ is positively homogeneous, in the sense that $R_\lambda(tu) = tR_\lambda(u)$ for any $t \geqslant 0$ and $u \in \widetilde{X}$. Also, the mapping $u \to au^{+} - bu^{-} : \widetilde{X} \to \widetilde{Y}$ is compact, and hence the operator $I - R_\lambda$ is compact. Now, letting $\widetilde{B}_r(c)$ denote the ball in $\widetilde{X}$ with radius r and centre c (and putting $\widetilde{B}_r := \widetilde{B}_r(0)$, for brevity), we see that the Leray-Schauder degree, $\deg(R_\lambda, \widetilde{B}_1, 0)$, is well-defined whenever λ is not a half-eigenvalue, see [5]. We will now relate the solvability properties of (3.5) and the degree $\deg(R_\lambda, \widetilde{B}_1, 0)$ to the location of λ relative to the set of half-eigenvalues. To state this precisely we introduce some further notation.

For each $k \geqslant 1$, let $\lambda_{k,\max} = \max\{\lambda_{k,+}, \lambda_{k,-}\}$, $\lambda_{k,\min} = \min\{\lambda_{k,+}, \lambda_{k,-}\}$, and define the open intervals

$$\Lambda_k^0 = \begin{cases} (\lambda_{k,\min}, \lambda_{k,\max}), & \text{if } \lambda_{k,\min} < \lambda_{k,\max}, \\ \emptyset, & \text{if } \lambda_{k,\min} = \lambda_{k,\max}, \end{cases}$$

$$\Lambda_k^1 = (\lambda_{k,\max}, \lambda_{k+1,\min}), \quad \Lambda_0^1 = (-\infty, \lambda_{1,\min}).$$

Intuitively, Theorem 3.1 says that when $a \neq b$ the term $au^{+} - bu^{-}$ in equation (3.1) 'splits apart' the linear eigenvalue λ_k into a pair of half-eigenvalues $\lambda_{k,\pm}(a, b)$, and the interval Λ_k^0 is the gap between these half-eigenvalues. It is possible for the half-eigenvalues $\lambda_{k,\pm}$ to coincide, so this gap may be empty. On the other hand, the inequality (3.2) says that half-eigenvalues with different values of k cannot coincide, so the interval Λ_k^1 between half-eigenvalues corresponding to k and $k+1$ is non-empty. Also, all these intervals are disjoint and their union comprises the whole of $\mathbb{R}$, except for the half-eigenvalues. Furthermore, by continuity, $\deg(R_\lambda, \widetilde{B}_1, 0)$ is constant on any of the intervals Λ_k^0, Λ_k^1.

Clearly, the intervals Λ_k^i, $i = 0, 1$, $k \geqslant 1$, depend on a, b, and when it is necessary to indicate this dependence explicitly we will write $\Lambda_k^i(a, b)$.

Theorem 3.9. (A) *If $\lambda \in \Lambda_k^1$, for some $k \geqslant 0$, then*:

(a) $\deg(R_\lambda, \widetilde{B}_1, 0) = (-1)^k$;
(b) *for any $h \in \widetilde{Y}$, equation* (3.5) *has a solution $u \in \widetilde{X}$.*

(B) *If $\lambda \in \Lambda_k^0$, for some $k \geqslant 1$, then*:

(a) $\deg(R_\lambda, \widetilde{B}_1, 0) = 0$;
(b) *there exists $h \in \widetilde{Y}$ such that equation* (3.5) *has no solution*;
(c) *there exists $h_b \in \widetilde{Y}$ such that, for any $h \in \widetilde{B}_1(h_b)$,* (3.5) *has at least two solutions.*

Remark 3.10. For the separated problem, similar results to those of Theorem 3.9 are proved in [10, Theorem 1.4] (for constant coefficients) and in [11, Theorem 5.1]

(for variable coefficients). The theorem shows that when λ is not a half-eigenvalue then a 'nonlinear Fredholm alternative' holds for (3.5), in the sense that either:

(a) there exists a solution u for all $h \in \widetilde{Y}$,

or

(b) there is no solution for some $h \in \widetilde{Y}$ and at least two solutions for other $h \in \widetilde{Y}$.

Such an interpretation was described in, for instance [10, Corollary 6.1].

Remark 3.11. The above results can be regarded as a generalization, to the half-linear, multi-point problem, of standard results from the linear spectral theory of Sturm-Liouville problems with separated boundary conditions. When $a = b$ the problem reduces to the linear case, so Theorem 3.9 also covers the linear, multi-point problem. Of course, in the linear case $\lambda_{k,\min} = \lambda_{k,\max}$ for all $k \geqslant 1$, so the intervals Λ_k^0 are empty and part (B) of Theorem 3.9 has no analogue. Furthermore, the degree $\deg(R_\lambda, \widetilde{B}_1, 0)$ changes by 2 as λ crosses a linear eigenvalue, which can be regarded, heuristically, as crossing two coincident half-eigenvalues, each of which contributes a change of 1.

The solvability and non-solvability results in Theorem 3.9 will be extended to the general nonlinear problem (1.1), (1.2), in Theorem 7.1 below

3.3. **The Fučík spectrum.** Using the above results we can now construct the Fučík spectrum Σ_F. We merely give a brief description here.

For $k \geqslant 1$, $\nu \in \{\pm\}$ and $\theta \in (0, \pi/2)$, let

$$\hat{\boldsymbol{r}}(\theta) := (\sin\theta, \cos\theta), \quad \lambda_{k,\nu}(\theta) := \lambda_{k,\nu}(\sin\theta, \cos\theta),$$

and define the C^1 curve

$$\sigma_{F,k,\nu} := \{\lambda_{k,\nu}(\theta)\hat{\boldsymbol{r}}(\theta) : \theta \in (0, \pi/2)\}.$$

It is clear from the preceding results that

$$\Sigma_F = \bigcup_{k,\nu} \sigma_{F,k,\nu}.$$

Some of the standard properties of the Fučík spectrum can easily be deduced from this characterisation. In particular, for each $k \geqslant 2$, $(\lambda_k, \lambda_k) \in \sigma_{F,k,\pm}$ (that is, the curves $\sigma_{F,k,\pm}$ intersect the diagonal line $\{(x,x) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ at the point (λ_k, λ_k)), and curves corresponding to different values of k do not intersect. Also, it follows readily from the Sturm comparison theorem that

$$\lim_{\theta \to 0} \lambda_{k,\pm}(\theta) = \lim_{\theta \to \pi/2} \lambda_{k,\pm}(\theta) = \infty,$$

so that the curves $\sigma_{F,k,\pm}$ have horizontal and vertical asymptotes. Furthermore, we can obtain analogues of the solvability properties in parts (A) and (B) of Theorem 3.9 in the gaps (in the plane $\mathbb{R}^2$) between the pairs of curves $\sigma_{F,k,\pm}$, or between the curves with consecutive values of k — for more details of such solvability properties in the Fučík setting see, for example, [12] or any of the other cited references dealing with the Fučík spectrum.

Another standard property of the Fučík spectrum is that the curves $\sigma_{F,k,\pm}$ are monotonically decreasing in $\mathbb{R}^2$. This is not so easy to prove here, with the parametrisation of the curves in terms of the angle θ. Since the main interest of the

Fučík spectrum is in obtaining solvability criteria, and we have obtained such criteria in Theorem 3.9, there seems little need to investigate the geometrical properties of Σ_F any further here.

The Fučík spectrum was first introduced by Fučík in [8], and the paper [4] contains a comprehensive investigation of this spectrum and its application to jumping nonlinearity problems. Many of the results here have analogues in [4] and, in varying degrees of generality, in many other papers since then (for separated problems).

4. Solutions with a single boundary condition

In this section we will construct solutions of equation (1.5) satisfying a single, multi-point boundary condition. For notational convenience in constructing such solutions, from now on (in this section and later) we will write $\lambda = s^2$, $a = \gamma_+^2$, $b = \gamma_-^2$ (given our hypothesis that $\lambda, a, b > 0$ this is possible), and we consider the problem

$$-u'' = s^2(\gamma_+^2 u^+ - \gamma_-^2 u^-), \quad \text{on } \mathbb{R}, \tag{4.1}$$

$$u(\eta_0) = \sum_{i=1}^m \alpha_i u(\eta_i), \tag{4.2}$$

for arbitrary $\alpha \in \mathcal{A}_+^m$, $\eta_0 \in \mathbb{R}$ and $\eta \in \mathbb{R}^m$. We can regard (4.1), (4.2) as a 'multi-point, initial value problem', and we will prove the following existence and 'uniqueness' result for this problem.

Theorem 4.1. *For fixed $s, \gamma_\pm > 0$, $m \geqslant 1$, $\alpha \in \mathcal{A}_+^m$, $\eta_0 \in \mathbb{R}$ and $\eta \in \mathbb{R}^m$, there exist functions $\psi_\pm \in C^2(\mathbb{R})$ such that $\pm\psi_\pm'(\eta_0) > 0$ and the set of solution of* (4.1), (4.2)*, has the form*

$$\{C_+\psi_+ : C_+ \geqslant 0\} \cup \{C_-\psi_- : C_- \geqslant 0\}.$$

Remark 4.2. Theorem 4.1 shows that if $\gamma_+ \neq \gamma_-$ then, in general, the solution set of (4.1), (4.2), consists of two 'half-rays' spanned by the functions $\psi_\pm$. However, if $\gamma_+ = \gamma_-$ (so that the problem is linear), then the solution set must be a linear subspace, so we have $\psi_+ = \psi_-$ and the solution set has the form $\{C\psi_+ : C \in \mathbb{R}\}$.

Proof. Define $\Psi \in C^2(\mathbb{R})$ to be the solution of the initial value problem

$$\begin{aligned} -\Psi'' &= \gamma_+^2\Psi^+ - \gamma_-^2\Psi^-, \\ \Psi(0) &= 0, \quad \Psi'(0) = 1. \end{aligned} \tag{4.3}$$

Clearly, Ψ has only simple zeros and, on any interval where $\pm\Psi > 0$, it satisfies the equation $-\Psi'' = \gamma_\pm^2\Psi$, so the graph of Ψ consists of a succession of positive and negative, sinusoidal 'bumps'. Thus, Ψ has the form

$$\Psi(x) = \pm\frac{1}{\gamma_\pm}\sin(\gamma_\pm x - \tau_\pm(x)), \quad \text{when } \pm\Psi(x) > 0, \tag{4.4}$$

where $\tau_\pm(x)$ is defined by

$$\gamma_\pm^{-1}\tau_\pm(x) := \max\{z \leqslant x : \Psi(z) = 0\},$$

that is, $\gamma_\pm^{-1}\tau_\pm(x)$ is the zero of Ψ immediately below x. Also, Ψ is periodic, with period

$$p_\Psi := \frac{\pi}{\gamma_+} + \frac{\pi}{\gamma_-}.$$

Next, for any $(s,\delta) \in (0,\infty) \times \mathbb{R}$ we define $w(s,\delta) \in C^2(\mathbb{R})$ by

$$w(s,\delta)(x) := \Psi(sx - \delta), \quad x \in \mathbb{R}.$$

It can easily be verified that any solution of (4.1) must have the form $Cw(s,\delta)$, for some $C \geqslant 0$ and $(s,\delta) \in (0,\infty) \times \mathbb{R}$, and $w(s,\delta)$ satisfies the boundary condition (4.2) if and only if

$$\Gamma(s,\delta,\alpha) := w(s,\delta)(\eta_0) - \sum_{i=1}^{m} \alpha_i w(s,\delta)(\eta_i) = 0. \tag{4.5}$$

Thus, it suffices to consider the set of solutions of (4.5). Clearly, the function $\Gamma : (0,\infty) \times \mathbb{R} \times \mathcal{A}_+^m \to \mathbb{R}$ is C^2, and we will denote the partial derivatives of Γ with respect to s and δ by Γ_s and Γ_δ.

Lemma 4.3. *For any* $(s,\delta,\alpha) \in (0,\infty) \times \mathbb{R} \times \mathcal{A}_+^m$,

$$\Gamma(s,\delta,\alpha) = 0 \implies \Gamma_\delta(s,\delta,\alpha) \neq 0.$$

Proof. Suppose that, for some $(s,\delta,\alpha) \in (0,\infty) \times \mathbb{R} \times \mathcal{A}_+^m$,

$$\Gamma(s,\delta,\alpha) = \Gamma_\delta(s,\delta,\alpha) = 0. \tag{4.6}$$

Suppose also, without loss of generality, that

$$w(s,\delta)(\eta_0) \geqslant 0, \quad w(s,\delta)(\eta_i) \geqslant 0, \quad 1 \leqslant i \leqslant p, \quad w(s,\delta)(\eta_i) < 0, \quad p < i \leqslant m.$$

Then, by (4.4), we can rewrite (4.6) as

$$S_0 = \sum_{i=1}^{p} \alpha_i S_i - \sum_{i=p+1}^{m} \alpha_i \frac{\gamma_+}{\gamma_-} S_i, \quad C_0 = \sum_{i=1}^{m} \alpha_i C_i, \tag{4.7}$$

where

$$S_i = \begin{cases} \sin(\gamma_+(s\eta_i - \delta) - \tau_+(s\eta_i - \delta)), & 0 \leqslant i \leqslant p, \\ \sin(\gamma_-(s\eta_i - \delta) - \tau_-(s\eta_i - \delta)), & p < i \leqslant m, \end{cases}$$

and the terms C_i, $0 \leqslant i \leqslant m$, are defined similarly, by replacing sin with cos. Now, by (4.7), and the fact that $S_i \geqslant 0$, $i = 0, \ldots, m$,

$$\begin{aligned} 1 = S_0^2 + C_0^2 &\leqslant S_0 \sum_{i=1}^{p} \alpha_i S_i + |C_0| \sum_{i=1}^{m} \alpha_i |C_i| \leqslant \sum_{i=1}^{m} \alpha_i \big(S_0 S_i + |C_0||C_i|\big) \\ &\leqslant \sum_{i=1}^{m} \alpha_i \big(S_0^2 + C_0^2\big)^{1/2} \big(S_i^2 + C_i^2\big)^{1/2} < 1, \end{aligned}$$

which shows that (4.6) cannot hold. $\square$

Lemma 4.4. *For any* $(s,\delta,\alpha) \in (0,\infty) \times \mathbb{R} \times \mathcal{A}_+^m$,

$$\Gamma(s,\delta,\alpha) = 0 \implies w(s,\delta)'(\eta_0) \neq 0.$$

Proof. Suppose that, for some $(s,\delta,\alpha) \in (0,\infty) \times \mathbb{R} \times \mathcal{A}_+^m$,

$$w(s,\delta)(\eta_0) = \sum_{i=1}^{m} \alpha_i w(s,\delta)(\eta_i) > 0 \quad \text{and} \quad w(s,\delta)'(\eta_0) = 0 \tag{4.8}$$

($w(s,\delta)(\eta_0) \neq 0$ since $w(s,\delta)$ is non-trivial, and the case $w(s,\delta)(\eta_0) < 0$ is similar). Then, by its repeating, sinusoidal form, the function $w(s,\delta)$ has a global max

at $x = \eta_0$, that is, $w(s,\delta)(x) \leqslant w(s,\delta)(\eta_0)$, for all $x \in \mathbb{R}$, so by (4.8) and the assumption that $\alpha \in \mathcal{A}_+^m$,

$$w(s,\delta)(\eta_0) = \sum_{i=1}^{m} \alpha_i w(s,\delta)(\eta_i) < w(s,\delta)(\eta_0).$$

This contradiction proves the lemma. □

For any $s > 0$ and $\alpha \in \mathcal{A}_+^m$, it follows from the above definitions that the function $\Gamma(s,\cdot,\alpha)$ is p_Ψ-periodic, so there are multiple zeros of (4.5) which do not yield distinct solutions of the problem (4.1), (4.2). To remove these additional zeros and to make the domain of δ compact, from now on we will regard δ as lying in the circle (which we denote by S^1) obtained from the interval $[0, p_\Psi]$ by identifying the points 0 and p_Ψ, and we then regard the domain of Γ as $(0,\infty) \times S^1 \times \mathcal{A}_+$ (clearly, Γ is still C^2).

For any fixed $s > 0$ the function $\Gamma(s,\cdot,0)$ has exactly two zeros $\delta_\pm(s,0) \in S^1$, and these zeros are simple and may be labelled so that $\pm w(s,\delta_\pm(s,0))'(\eta_0) > 0$. Hence, by a simple continuation argument, using Lemma 4.3 and the implicit function theorem, we see that $\Gamma(s,\cdot,\alpha)$ has exactly two zeros $\delta_\pm(s,\alpha) \in S^1$ for all $\alpha \in \mathcal{A}_+^m$, and these zeros depend continuously on α. Also, by Lemma 4.4,

$$\pm w(s,\delta_\pm(s,\alpha))'(\eta_0) = \pm w(s,\delta_\pm(s,0))'(\eta_0) > 0, \quad \alpha \in \mathcal{A}_+^m,$$

so setting $\psi_\pm := w(s,\delta_\pm(s,\alpha))$ completes the proof of Theorem 4.1. □

Remark 4.5. The variable s has been fixed throughout this section, but the above notation and Lemmas 4.3 and 4.4 will be required in Section 5, where s will vary.

5. Proof of Theorem 3.1 and Corollary 3.2

Since we have assumed that $a, b, \lambda > 0$ we may now rewrite the half-eigenvalue problem (3.1) in the form

$$-\Delta u = s^2(\gamma_+^2 u^+ - \gamma_-^2 u^-), \quad u \in X \tag{5.1}$$

(that is, we have put $\lambda = s^2$, $a = \gamma_+^2$, $b = \gamma_-^2$), and we may apply the constructions in Section 4 to (5.1). In particular, we continue to use the solution $w(s,\delta)$ of the differential equation (4.1) defined there. Substituting $w(s,\delta)$ into the boundary conditions (1.2) now shows that a number $\lambda = s^2$ is a half-eigenvalue of (5.1) if and only if the pair of equations

$$\Gamma^\pm(s,\delta,\alpha^\pm) := w(s,\delta)(\pm 1) - \sum_{i=1}^{m^\pm} \alpha_i^\pm w(s,\delta)(\eta_i^\pm) = 0, \tag{5.2}$$

is satisfied, for some $\delta \in \mathbb{R}$, and then $w(s,\delta)$ is a corresponding half-eigenfunction. Thus, we will prove the theorem by considering the set of solutions of (5.2). As for the function Γ in Section 4, we will regard the domains of the functions $\Gamma^\pm$ as $(0,\infty) \times S^1 \times \mathcal{A}_+$. Again, it is clear that these functions are C^2 on this domain.

The following proposition now proves the existence and uniqueness of the half-eigenvalues.

Proposition 5.1. *For each $k \geqslant 1$, $\nu \in \{\pm\}$ and $\boldsymbol{\alpha} \in \mathcal{A}_+$, there is exactly one solution $(s_{k,\nu}(\boldsymbol{\alpha}), \delta_{k,\nu}(\boldsymbol{\alpha})) \in (0,\infty) \times S^1$ of (5.2) such that $w(s_{k,\nu}(\boldsymbol{\alpha}), \delta_{k,\nu}(\boldsymbol{\alpha})) \in T_{k,\nu}$. There are no other solutions of (5.2) in $(0,\infty) \times S^1$.*

Proof. When $\boldsymbol{\alpha} = \mathbf{0}$ the problem (3.1) is a constant coefficient, half-eigenvalue problem with separated (Dirichlet) boundary conditions, so it is elementary to explicitly construct the half-eigenvalues and corresponding half-eigenfunctions, which we will write as $\lambda^{\mathbf{0}}_{k,\pm} = (s^{\mathbf{0}}_{k,\pm})^2$, $u^{\mathbf{0}}_{k,\pm} = w(s^{\mathbf{0}}_{k,\pm}, \delta^{\mathbf{0}}_{k,\pm}) \in T_{k,\pm}$, $k \geqslant 1$, for suitable $s^{\mathbf{0}}_{k,\pm}$, $\delta^{\mathbf{0}}_{k,\pm}$, see [4] or the proof of [7, Theorem 11.5] for the details. This yields the following lemma (which we state for reference).

Lemma 5.2. *When $\boldsymbol{\alpha} = \mathbf{0}$, Proposition 5.1 holds with corresponding solutions $(s^{\mathbf{0}}_{k,\nu}, \delta^{\mathbf{0}}_{k,\nu})$, $k \geqslant 1$, $\nu \in \{\pm\}$.*

To extend Lemma 5.2 to the case $\boldsymbol{\alpha} \neq \mathbf{0}$ we require the following lemmas.

Lemma 5.3. *For any $(s, \delta, \boldsymbol{\alpha}) \in (0, \infty) \times \mathbb{R} \times \mathcal{A}_+$, and $\nu \in \{\pm\}$,*

$$\Gamma^\nu(s, \delta, \alpha^\nu) = 0 \implies \begin{cases} w(s, \delta)'(\nu) \neq 0, \\ \Gamma^\nu_s(s, \delta, \alpha^\nu)\, \Gamma^\nu_\delta(s, \delta, \alpha^\nu) \neq 0 \end{cases}$$

(where $w(s, \delta)'(\pm)$ means $w(s, \delta)'(\pm 1)$).

Proof. This follows immediately from Lemmas 4.3 and 4.4, and the definitions of Γ and $\Gamma^\pm$ (the proof that $\Gamma^\pm_s(s, \delta, \alpha) \neq 0$ is similar to the proof of Lemma 4.3, using the additional fact that $|\eta^\pm_i| \leqslant 1$, $i = 0, \ldots, m^\pm$, here). □

Lemma 5.4. *For any $(s, \delta, \boldsymbol{\alpha}) \in (0, \infty) \times \mathbb{R} \times \mathcal{A}_+$,*

$$\Gamma^+(s, \delta, \alpha^+) = \Gamma^-(s, \delta, \alpha^-) = 0 \implies w(s, \delta) \in T_{k,\nu}, \textit{ for some } k \geqslant 1 \textit{ and } \nu \in \{\pm\}.$$

Proof. By the definition of the sets $T_{k,\nu}$ and the form of $w(s, \delta)$, it suffices to show that $w(s, \delta)'(\pm 1) \neq 0$, but this follows from Lemma 5.3. □

The proof of Proposition 5.1 can now be completed by following the continuation argument in the proof of [9, Theorem 4.1], so we will merely outline the argument here. Lemmas 5.2, 5.3 and 5.4 above provide the necessary analogues, in the current setting, of Corollary 4.5, Lemma 4.6 and equation (4.6) in [9]. The argument is then similar to that in the proof of Theorem 4.1, except that we now have the pair of equations (5.2) to consider, rather than the single equation (4.5). However, as shown in the the proof of [9, Theorem 4.1], the above results enable us to apply the implicit function theorem at an arbitrary solution $(s, \delta, \boldsymbol{\alpha})$ of (5.2) (with $\boldsymbol{\alpha} = (\alpha^+, \alpha^-) \in \mathcal{A}_+$), so we may construct the entire set of zeros of (5.2), at an arbitrary $\boldsymbol{\alpha} \in \mathcal{A}_+$, by continuation away from the zeros at $\boldsymbol{\alpha} = \mathbf{0}$ (as given in Lemma 5.2). □

Proposition 5.1 has proved the existence and uniqueness of the half-eigenvalues $\lambda_{k,\nu}(\boldsymbol{\alpha}) := s^2_{k,\nu}(\boldsymbol{\alpha})$, with half-eigenfunctions $u_{k,\nu}(\boldsymbol{\alpha}) := w(s_{k,\nu}(\boldsymbol{\alpha}), \delta_{k,\nu}(\boldsymbol{\alpha}))$ for $\boldsymbol{\alpha} \in \mathcal{A}_+$. We will now prove that these half-eigenvalues are increasing, in the sense of inequality (3.2). To do this we first note that, by the explicit construction when $\boldsymbol{\alpha} = \mathbf{0}$, the half-eigenvalues $\lambda_{k,\nu}(\mathbf{0}) = \lambda^{\mathbf{0}}_{k,\nu}$ satisfy (3.2). Thus, by the continuation construction of $\lambda_{k,\nu}(\boldsymbol{\alpha})$, for general $\boldsymbol{\alpha} \in \mathcal{A}_+$, it suffices to show that $\lambda_{k,\nu}(\boldsymbol{\alpha}) \neq \lambda_{k+1,\nu'}(\boldsymbol{\alpha})$ for all $k \geqslant 1$, $\nu, \nu' \in \{\pm\}$ and $\boldsymbol{\alpha} \in \mathcal{A}_+$.

Suppose, on the contrary, that $\lambda_{k,\nu}(\boldsymbol{\alpha}) = \lambda_{k+1,\nu}(\boldsymbol{\alpha})$, for some such k, ν and $\boldsymbol{\alpha}$. Then, by Proposition 5.1 and the definition of the sets $T_{k,\nu}, T_{k+1,\nu}$,

$$\operatorname{sgn} u_{k,\nu}(\boldsymbol{\alpha})'(-1) = \operatorname{sgn} u_{k+1,\nu}(\boldsymbol{\alpha})'(-1),$$

so Theorem 4.1, together with the boundary condition (4.2) at $\eta_0 = -1$, shows that $u_{k,\nu}(\boldsymbol{\alpha}) = u_{k+1,\nu}(\boldsymbol{\alpha})$, which is a contradiction, since the sets $T_{k,\nu}, T_{k+1,\nu}$ are disjoint. Now suppose that $\lambda_{k,\nu}(\boldsymbol{\alpha}) = \lambda_{k+1,-\nu}(\boldsymbol{\alpha})$. Since, by definition, the derivative of a function in $T_{k,\nu}$ changes sign exactly k times in the interval $(-1,1)$, we see that in this case

$$\operatorname{sgn} u_{k,\nu}(\boldsymbol{\alpha})'(1) = \operatorname{sgn} u_{k+1,-\nu}(\boldsymbol{\alpha})'(1),$$

so Theorem 4.1, together with the boundary condition (4.2) at $\eta_0 = 1$, now shows that $u_{k,\nu} = u_{k+1,-\nu}$, which is again a contradiction. This proves that the half-eigenvalues are increasing.

Next, the fact that $\lim_{k\to\infty} \lambda_{k,\pm} = \infty$ is clear from the Sturm comparison theorem, so it only remain to prove that $\pm u_{1,\pm}$ is strictly positive on $(-1,1)$. To do this we observe that a half-eigenfunction which does not change sign is in fact an eigenfunction of a linear problem. Specifically, recalling Remark 3.5, it is shown in [15, Theorem 3.1] that the linear eigenfunction u_1 may be chosen to be strictly positive on $(-1,1)$ and $u_1 \in T_{1,+}$, so by the above uniqueness result for the half-eigenvalues,

$$\lambda_{1,+} = \lambda_1/a,\ u_{1,+} = u_1, \quad \lambda_{1,-} = \lambda_1/b,\ u_{1,-} = -u_1$$

(up to a positive scalar multiple of the eigenfunctions). This proves the desired positivity, and finally completes the proof of Theorem 3.1. □

The proof of Corollary 3.2 now follows immediately from the implicit function theorem and the continuation construction of the half-eigenvalues, using the fact that the functions $\Gamma^\pm$ are C^1 functions of the variables $(a,b,\boldsymbol{\alpha},\boldsymbol{\eta})$ (although this dependence was suppressed above).

6. Proof of Theorem 3.9

6.1. **Part (A).** Corollary 3.2 showed that the half-eigenvalues depend continuously on $\boldsymbol{\alpha} \in \mathcal{A}_+$, and for the duration of this proof we indicate this dependence explicitly by writing $\lambda_{k,\pm}(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathcal{A}_+$, for $k \geqslant 1$. Furthermore, for the duration of this proof, we indicate the dependence on $\boldsymbol{\alpha}$ of the space $\widetilde{X}$, and hence the ball $\widetilde{B}_1 \subset \widetilde{X}$ and the operator $R_\lambda : \widetilde{X} \to \widetilde{X}$, by writing $\widetilde{X}_{\boldsymbol{\alpha}}$, $\widetilde{B}_{1,\boldsymbol{\alpha}}$ and $R_{\lambda,\boldsymbol{\alpha}}$. We can readily construct a continuous family of bounded, linear isomorphisms $S_{\boldsymbol{\alpha}} : \widetilde{X}_{\mathbf{0}} \to \widetilde{X}_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{A}_+$, with $S_{\mathbf{0}}$ the identity on $\widetilde{X}_{\mathbf{0}}$. We now fix $k \geqslant 1$ and prove the result for arbitrary (fixed) $\boldsymbol{\alpha}^0 \in \mathcal{A}_+$ and $\lambda^0 \in (\lambda_{k,\max}(\boldsymbol{\alpha}^0), \lambda_{k+1,\min}(\boldsymbol{\alpha}^0))$.

By (3.2), we can choose a continuous function $\rho : [0,1] \to \mathbb{R}$ such that

$$\rho(t) \in (\lambda_{k,\max}(t\boldsymbol{\alpha}^0), \lambda_{k+1,\min}(t\boldsymbol{\alpha}^0)), \quad t \in [0,1], \qquad \rho(1) = \lambda^0, \tag{6.1}$$

and define $T_t : \widetilde{X}_{\mathbf{0}} \to \widetilde{X}_{\mathbf{0}}$, $t \in [0,1]$, by

$$T_t(u) := S^{-1}_{t\boldsymbol{\alpha}^0} R_{\rho(t),t\boldsymbol{\alpha}^0} S_{t\boldsymbol{\alpha}^0} u, \quad u \in \widetilde{X}_{\mathbf{0}}.$$

Clearly, $T_t(u)$ depends continuously on $(t,u) \in [0,1] \times \widetilde{X}_{\mathbf{0}}$. Also, by (6.1) and the definition of T_t, for each $t \in [0,1]$ there is no non-trivial solution of the equation $T_t(u) = 0$, so by standard properties of the degree,

$$\deg(R_{\lambda^0}, \widetilde{B}_{1,\boldsymbol{\alpha}^0}, 0) = \deg(T_1, \widetilde{B}_{1,\mathbf{0}}, 0) = \deg(T_0, \widetilde{B}_{1,\mathbf{0}}, 0) = \deg(R_{\rho(0),\mathbf{0}}, \widetilde{B}_{1,\mathbf{0}}, 0).$$

Now, recalling that when $\boldsymbol{\alpha} = \mathbf{0}$ the problem is a Dirichlet problem, it is shown in [11, Theorem 5.5] (which deals with the separated problem) that

$$\deg(R_{\rho(0),\mathbf{0}}, \widetilde{B}_{1,\mathbf{0}}, 0) = (-1)^k,$$

which proves part (A)-(a). Part (A)-(b) now follows from part (A)-(a), the positive homogeneity of the operator R_{λ^0}, and standard properties of the degree, see parts (D4) and (D5) of Theorem 8.2 in [5]. Finally, a simple modification of the above argument also proves the result for the case $k = 0$ (simply by omitting any reference to $\lambda_{0,\max}$ in this case).

6.2. **Part (B).** We first prove part (B)-(b) — part (B)-(a) then follows immediately since, by the above argument proving part A-(b), the existence of h for which equation (3.5) has no solution shows that $\deg(R_\lambda, \widetilde{B}_1, 0) = 0$.

For any $s > 0$, we set $m = m^-$, $\eta_0 = -1$, $\eta = \eta^-$ and $\alpha = \alpha^-$, and use the results of Section 4 to construct the corresponding numbers $\delta_\pm(s, \alpha^-)$, and functions $\psi_\pm = w(s, \delta_\pm(s, \alpha^-))$ which satisfy (1.5) and the boundary condition (1.2) at -1 (as in Theorem 4.1). Thus, recalling the functions $\Gamma^\pm$ defined in (5.2), in the proof of Theorem 3.1, we see that $\Gamma^-(s, \delta_\pm(s, \alpha^-), \alpha^-) = 0$, and $\lambda = s^2$ is a half-eigenvalue iff

$$B(\lambda) := \Gamma^+(s, \delta_-(s, \alpha^-), \alpha^+)\Gamma^+(s, \delta_+(s, \alpha^-), \alpha^+) = 0.$$

Also, the results in the proof of Theorem 3.1 show that if, for some $k \geqslant 1$, $\lambda_{k,-} \neq \lambda_{k,+}$ then the sign of $B(\lambda)$ changes as λ crosses the half-eigenvalues $\lambda_{k,\pm}$.

Proposition 6.1. *If $B(\lambda) > 0$ then there exists $h \in \widetilde{Y}$ such that equation* (3.5) *has no solution.*

Proof. Suppose further that

$$\Gamma^+(s, \delta_\pm(s, \alpha^-), \alpha^+) > 0 \tag{6.2}$$

(the other case is similar). In particular, (6.2) implies that $\psi_\pm(1) > 0$. We first construct a suitable function h. Consider the initial value problem

$$\begin{aligned} -v'' &= av + \lambda v - 1, \\ v(x_l) &= 0, \quad v'(x_l) \geqslant 0, \end{aligned} \tag{6.3}$$

for arbitrary $x_l \in (-1, 1)$. It can easily be shown that there exists a sufficiently small $\delta > 0$ such that if $x_l \in [1 - \delta, 1)$ then any solution v of (6.3) has no zero in $(x_l, 1]$. We can also choose δ sufficiently small that $\eta_i^\nu < 1 - \delta$, for $1 \leqslant i \leqslant m^\nu$, $\nu \in \{\pm\}$, and $\psi_\pm(x) \neq 0$ for $x \in [1 - \delta, 1]$. Let $x_0 = 1 - \delta$, and define

$$h(x) = \begin{cases} 0, & x \in [0, x_0), \\ -1, & x \in [x_0, 1]. \end{cases}$$

Now, for any $\gamma \in \mathbb{R}$, let $\Phi_{\gamma,0} := |\gamma|\psi_{\operatorname{sgn}\gamma}$ (so $\Phi_{\gamma,0}$ satisfies (1.5) and the boundary condition (1.2) at -1) and let $\Phi_{\gamma,h}$ denote the solution of the differential equation corresponding to (3.5) satisfying $\Phi_{\gamma,h} \equiv \Phi_{\gamma,0}$ on $[-1, x_0]$. Any solution of (3.5) must be of the form $\Phi_{\gamma,h}$, for some $\gamma \in \mathbb{R}$, and $\Phi_{\gamma,h}$ is a solution of (3.5) if and only if it satisfies the boundary condition (1.2) at 1. We will show that this cannot happen for any $\gamma \in \mathbb{R}$.

Clearly, if $\gamma = 0$ then $\Phi_{\gamma,0} \equiv 0$, while if $\gamma \neq 0$ then $\Phi_{\gamma,0}$ has no zeros on $[x_0, 1]$. Furthermore, from the form of h,

$$\Phi_{\gamma,0}(x_0) = \Phi_{\gamma,h}(x_0), \quad \Phi'_{\gamma,0}(x_0) = \Phi'_{\gamma,h}(x_0),$$

and $\Phi_{\gamma,h}(x) > \Phi_{\gamma,0}(x)$, for sufficiently small $x - x_0 > 0$. In fact, we have the following result.

Lemma 6.2. *For any $\gamma \in \mathbb{R}$, $\Phi_{\gamma,h} > \Phi_{\gamma,0}$, on $(x_0, 1]$.*

Proof. Suppose the contrary, and let x_1 be the first zero of $\Phi_{\gamma,h} - \Phi_{\gamma,0}$ in $(x_0, 1]$. Then $\Phi_{\gamma,h} - \Phi_{\gamma,0} > 0$ in (x_0, x_1), and $\Phi'_{\gamma,h}(x_1) - \Phi'_{\gamma,0}(x_1) \leqslant 0$. Now suppose that $\Phi_{\gamma,0} \geqslant 0$ on $(x_0, 1)$, and define

$$W = \Phi'_{\gamma,h}\Phi_{\gamma,0} - \Phi_{\gamma,h}\Phi'_{\gamma,0}.$$

Then $W(x_0) = 0$, and by equation (3.5), $W' > 0$ on (x_0, x_1), so that $W > 0$ on $(x_0, x_1]$. On the other hand,

$$W(x_1) = \Phi_{\gamma,0}(x_1)\big(\Phi'_{\gamma,h}(x_1) - \Phi'_{\gamma,0}(x_1)\big) \leqslant 0,$$

and this contradiction deals with the case $\Phi_{\gamma,0} \geqslant 0$ on $(x_0, 1)$. Now suppose that $\Phi_{\gamma,0} < 0$ on $(x_0, 1)$. If $\Phi_{\gamma,h} < 0$ on (x_0, x_1) then a similar argument deals with this case so we suppose that $\Phi_{\gamma,h}$ changes sign on (x_0, x_1). But, by the choice of δ, the function $\Phi_{\gamma,h}$ can have at most one zero in $(x_0, 1]$, so we again obtain $\Phi_{\gamma,h} > \Phi_{\gamma,0}$, on $(x_0, 1]$, which completes the proof of Lemma 6.2. □

Now, for any $\gamma \in \mathbb{R}$, combining (6.2) and Lemma 6.2 yields

$$\begin{aligned}\Phi_{\gamma,h}(1) > \Phi_{\gamma,0}(1) = |\gamma|\psi_{\operatorname{sgn}\gamma}(1) &\geqslant \sum_{i=1}^{m^+} \alpha_i^+ |\gamma| \psi_{\operatorname{sgn}\gamma}(\eta_i^+) \\ &= \sum_{i=1}^{m^+} \alpha_i^+ \Phi_{\gamma,0}(\eta_i^+) = \sum_{i=1}^{m^+} \alpha_i^+ \Phi_{\gamma,h}(\eta_i^+),\end{aligned}$$

which completes the proof of Proposition 6.1. □

Now, combining Proposition 6.1 with part (A) shows that

$$\bigcup_{k\geqslant 1} \Lambda_k^0 = \{\lambda : B(\lambda) > 0\}, \quad \bigcup_{k\geqslant 0} \Lambda_k^1 = \{\lambda : B(\lambda) < 0\},$$

and hence completes the proof of part (B)-(b) of Theorem 3.9.

To prove part (B)-(c) we require the following two lemmas. The first lemma shows that the operator $R_\lambda : \widetilde{X} \to \widetilde{X}$ has a Fréchet derivative at certain points $u \in \widetilde{X}$, which we denote by $D_u R_\lambda(u)$, and gives the form of this derivative. For any $u \in \widetilde{X}$ we denote the characteristic functions of the sets $\{x \in [-1, 1] : \pm u(x) > 0\}$ by $\chi_{u^\pm}$.

Lemma 6.3. *If $u \in \widetilde{X}$ has only simple zeros in $[-1, 1]$ then, for any $\lambda > 0$, the operator R_λ is Fréchet differentiable at u, with $D_u R_\lambda(u)$ given by*

$$D_u R_\lambda(u)v = v + \lambda\widetilde{\Delta}^{-1}\big((a\chi_{u^+} + b\chi_{u^-})v\big), \quad v \in \widetilde{X},$$

Proof. The proof is similar to the proof of [10, Lemma 3.1]. □

Lemma 6.4. *If $\lambda \in \Lambda_k^0$, for some $k \geqslant 1$, then there exists $\omega = \omega(\lambda) \in \widetilde{X}$ such that $D_u R_\lambda(\omega)$ is non-singular.*

Proof. Choosing $\omega_1 \in \widetilde{X}$ such that $\omega_1 > 0$, it follows from Lemma 6.3 that

$$D_u R_\lambda(\omega_1)v = v + \lambda\widetilde{\Delta}^{-1}(av), \quad v \in \widetilde{X},$$

and hence $D_u R_\lambda(\omega_1)$ is non-singular (so ω_1 suffices for the result) unless the equation

$$-\widetilde{\Delta}v = \lambda a v, \quad v \in \widetilde{X}, \tag{6.4}$$

has a non-trivial solution v.

Suppose that (6.4) has a non-trivial solution v_1. Choose x_j, $j = 1, 2$, such that $\eta_i^\nu < x_1 < x_2 \leqslant 1$ for all $i = 1, \dots, m^\nu$, $\nu \in \{\pm\}$, and choose $\omega_2 \in \widetilde{X}$, such that ω_2 has a simple zero at each x_j, and $\omega_2 > 0$ on $[-1, x_1) \cup (x_2, 1]$, $\omega_2 < 0$ on (x_1, x_2). We now have

$$D_u R_\lambda(\omega_2)v = v + \lambda \widetilde{\Delta}^{-1}(a\chi_{\omega_2^+} + b\chi_{\omega_2^-})v, \quad v \in \widetilde{X}$$

(note that since $\lambda \in \Lambda_k^0$ we must have $a \neq b$), and now $D_u R_\lambda(\omega_2)$ is non-singular (so ω_2 suffices for the result) unless the equation

$$-\widetilde{\Delta}v = \lambda(a\chi_{\omega_2^+} + b\chi_{\omega_2^-})v, \quad v \in \widetilde{X}, \tag{6.5}$$

has a non-trivial solution v.

Suppose that (6.5) has a non-trivial solution v_2. Given the choice of x_1, x_2, it follows from the boundary condition (1.2) at $x = -1$ and Remark 4.2, together with equations (6.4) and (6.5), that we may suppose that $v_1 \equiv v_2$, on $[-1, x_1]$, and hence, by the boundary condition (1.2) at $x = 1$, we must have

$$v_1(1) = v_2(1). \tag{6.6}$$

We now show that there exist x_1, x_2 such that (6.6) is false.

Lemma 6.5. *If $x_2 = 1$ and x_1 is sufficiently close to 1 then $v_1(1) \neq v_2(1)$.*

Proof. Suppose that $v_1(1) = v_2(1)$. Then we may choose x_1 sufficiently close to 1 that v_1, v_2 are non-zero on $[x_1, 1)$, and satisfy

$$-v_1'' = \lambda a v_1, \quad -v_2'' = \lambda b v_2$$
$$v_1(x_1) = v_2(x_1), \quad v_1'(x_1) = v_2'(x_1), \quad v_1(1) = v_2(1).$$

Since $a \neq b$, a slight modification of the proof of Theorem 1.2 in Chapter 8 of [3] (the Sturm comparison theorem) now shows that this is impossible. □

Returning to the proof of Lemma 6.4, we choose x_1 as in Lemma 6.5, and we can then choose $x_2 < 1$ so that $v_1(1) \neq v_2(1)$ (by continuous dependence of v_2 on x_2). This shows that, for this choice of x_1, x_2, (6.6) is false, and hence $D_u R_\lambda(\omega_2)$ is non-singular. This completes the proof of Lemma 6.4. □

We now proceed with the proof of part (B)-(c) of Theorem 3.9. Let $h_b := -\widetilde{\Delta} R_\lambda(\omega) \in \widetilde{Y}$, where ω is as in Lemma 6.4. Then ω is an isolated solution of the equation $R_\lambda(u) = -\widetilde{\Delta}^{-1} h_b$, with index ± 1. Hence, by continuity properties of the degree and part (B)-(a), there exists sufficiently small numbers r_1, r_2, $r_3 > 0$ such that, for any $h \in \widetilde{B}_{r_3}(h_b)$,

$$\deg(R_\lambda, \widetilde{B}_1(0), -r_1\widetilde{\Delta}^{-1}(h)) = 0, \quad \deg(R_\lambda, \widetilde{B}_{r_2}(r_1\omega), -r_1\widetilde{\Delta}^{-1}(h)) \neq 0,$$

and so the equation $R_\lambda(u) = -r_1\widetilde{\Delta}^{-1}(h)$ has solutions in the balls $\widetilde{B}_{r_2}(r_1\omega)$ and $\widetilde{B}_1(0) \setminus \widetilde{B}_{r_2}(r_1\omega)$ (we assume that the numbers r_i are sufficiently small that $\widetilde{B}_{r_2}(r_1\omega) \subset \widetilde{B}_1(0)$). The result as stated in the theorem now follows by scaling and using the positive homogeneity of R_λ. This proves part (B)-(c), and so completes the proof of Theorem 3.9.

7. Non-linear problems

In this section we consider the solvability properties of the problem (1.1), (1.2), which we rewrite as

$$-\Delta u = f(u) + h, \quad u \in \widetilde{X} \tag{7.1}$$

(here, for any $u \in C^0[-1,1]$ we let $f(u) \in C^0[-1,1]$ denote the function $f(u(x))$, $x \in [-1,1]$). We have the following analogue of Theorem 3.9.

Theorem 7.1. (A) *If $1 \in \Lambda_k^1(f_\infty, f_{-\infty})$, for some $k \geqslant 0$, then for any $h \in \widetilde{Y}$ equation* (7.1) *has a solution $u \in \widetilde{X}$.*
(B) *If $1 \in \Lambda_k^0(f_\infty, f_{-\infty})$, for some $k \geqslant 1$, then there exists h_0, $h_2 \in \widetilde{Y}$ such that if $h = h_0$ then equation* (7.1) *has no solution, while if $h = h_2$ then equation* (7.1) *has two solutions.*

Proof. The proof of part (A) and the zero solution assertion in part (B) is similar to the proof of parts (iii) and (iv) of [4, Theorem 5], using the results of Theorem 3.9 above, while the proof of the two solutions assertion is a slight extension of the above proof of the corresponding result in Theorem 3.9. □

The solvability results stated in Theorem 7.1 were obtained in [4, Theorem 5] for the equation (7.1) with Dirichlet boundary conditions. These results were extended to more general, separated boundary conditions and variable coefficients in [11, Theorem 6.1]. Similar results for other separated problems have been obtained in many other papers (see, for example, the references in [11, 12]). In many of these papers the results have been expressed in terms of hypotheses on the location of the point $(f_\infty, f_{-\infty}) \in \mathbb{R}^2$ relative to the Fučík spectrum of the problem. The hypotheses in Theorem 7.1 ensure that $\lambda = 1$ is not a half-eigenvalue, and so $(f_\infty, f_{-\infty})$ is not in the Fučík spectrum. In fact, in the constant coefficient case considered here these hypotheses are equivalent to the usual conditions on the location of the point $(f_\infty, f_{-\infty})$ relative to the Fučík spectrum (see [11, 12] for a more detailed discussion of the relationship between the half-eigenvalues and the Fučík spectrum in the separated problem, which applies equally well here). That is, the Fučík spectrum and the half-eigenvalues are equivalent concepts here. However, it was convenient to prove Theorem 3.1, in particular, using the half-eigenvalue parameter λ. Furthermore, as mentioned in the introduction, it is shown in [11, 12] that in the general, separated, variable coefficient case the results obtained using half-eigenvalues are stronger than those obtained using the Fučík spectrum approach, so it seemed preferable to state our results in terms of half-eigenvalues rather than the Fučík spectrum.

8. Global bifurcation and nodal solutions

In this final section we suppose that

$$f(0) = 0, \quad f_0 := \lim_{s \to 0} \frac{f(s)}{s} > 0 \tag{8.1}$$

(we assume that this limit exists and is finite), and we briefly describe a Rabinowitz-type global bifurcation theorem and then obtain nodal solutions of (7.1) with $h = 0$. Given the preceding results, these results are now relatively standard so we omit most of the details.

8.1. **Global bifurcation.** We first briefly consider the bifurcation problem,

$$-\Delta(u) = \lambda f(u), \quad (\lambda, u) \in \mathbb{R} \times X. \tag{8.2}$$

Clearly, by (8.1), $u \equiv 0$ is a solution of (8.2) for any $\lambda \in \mathbb{R}$; such solutions will be called *trivial*. The following Rabinowitz-type global bifurcation theorem for nontrivial solutions of (8.2) was proved in [15, Theorem 6.2] (recall the eigenvalues λ_k introduced in Remark 3.5).

Theorem 8.1. *For each $k \geqslant 1$ there exist closed, connected sets $\mathcal{C}_k^{\pm} \subset (0,\infty) \times X$ of solutions of* (8.2) *with the properties*:

(a) $(\lambda_k/f_0, 0) \in \mathcal{C}_k^{\pm}$;
(b) $\mathcal{C}_k^{\pm} \backslash \{(\lambda_k/f_0, 0)\} \subset (0,\infty) \times T_k^{\pm}$;
(c) $\mathcal{C}_k^{\pm}$ *is unbounded in* $(0,\infty) \times Y$.

8.2. **Nodal solutions.** We now search for nodal solutions of the problem

$$-\Delta u = f(u), \quad u \in X \tag{8.3}$$

(that is, solutions u lying in specific sets $T_{k,\nu}$). We suppose through this section that

$$sf(s) > 0, \quad s \neq 0. \tag{8.4}$$

Theorem 8.2. *If* (8.4) *holds and*

$$(\lambda_k/f_0 - 1)(\lambda_{k,\nu}(f_\infty, f_{-\infty}) - 1) < 0, \tag{8.5}$$

for some $k \geqslant 1$ and ν, then (8.3) *has a solution $u \in T_{k,\nu}$.*

Proof. The proof is essentially the same as the proof of [14, Theorem 7.1] (although the half-eigenvalues used in [14] were for problems of the form (3.3)), so we merely sketch it here.

By Theorem 8.1 there is a continuum, $\mathcal{C}_{k,\nu}$, of solutions of (8.2), bifurcating from $(\lambda_k/f_0, 0)$, and by the argument in [14] it can be shown that $\mathcal{C}_{k,\nu}$ 'meets $(\lambda_{k,\nu}(f_\infty, f_{-\infty}), \infty)$' (more precisely, there exists a sequence $(\mu_n, u_n) \in \mathcal{C}_{k,\nu}$, $n = 1, 2, \ldots$, such that $\mu_n \to \lambda_{k,\nu}(f_\infty, f_{-\infty})$, $|u_n|_0 \to \infty$). Since $\mathcal{C}_{k,\nu}$ is connected, it now follows from (8.5) that $\mathcal{C}_{k,\nu}$ must intersect the hyperplane $\{1\} \times T_{k,\nu}$ at a point $(1, u)$, and hence $u \in T_{k,\nu}$ is a solution of (8.3). □

Remarks 8.3. Nodal solutions for similar problems have been obtained previously, in several papers:

- [6], [13] and [14] considered one separated and one multi-point Dirichlet boundary condition;
- [15] considered two multi-point Dirichlet boundary conditions;
- [16] considered two multi-point Neumann boundary conditions, or a mixture of Dirichlet and Neumann conditions.

NB the papers [6], [15], [16] dealt with equations involving the p-Laplacian.
We briefly summarize the result obtained in these papers.

(a) Theorem 5.7 in [6] obtained similar nodal solutions to Theorem 8.2 above, but the limits (1.3) were not assumed to exist (and the half-eigenvalues had not been obtained) in [6], so instead of the above condition (8.5) a condition involving certain lim sup's and lim inf's of $f(s)/s$, as $s \to \infty$, was used. When applicable, condition (8.5) is a weaker hypothesis than the conditions in [6, Theorem 5.7].

(b) Theorem 7.1 in [14] is similar to Theorem 8.2 above (using half-eigenvalues),

but only considers a 3-point boundary condition, at one end point.

(c) When $f_\infty = f_{-\infty}$ the nonlinearity f is 'asymptotically linear'. Theorem 8.2 clearly applies to such problems, and the half-eigenvalues $\lambda_{k,\nu}(f_\infty, f_{-\infty})$ reduce to λ_k/f_∞. Such problems were considered in all the above cited papers.

(d) The 'superlinear' case $f_\infty = f_{-\infty} = \infty$ has also been considered, see [6, Theorem 5.5] and [13, Theorem 5.4]. This case does not involve half-eigenvalues, and the proofs of these results extend readily to the present setting, so we simply state the corresponding result here, without proof.

Theorem 8.4. *Suppose that $f_\infty = f_{-\infty} = \infty$. If (8.4) holds and $\lambda_k/f_0 > 1$, for some $k_0 \geqslant 1$, then for each $k \geqslant k_0$, (8.3) has a solution $u \in T_{k,\pm}$.*

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